# Remarks on the Charged, Magnetized Tomimatsu-Sato $\delta = 2$ Solution

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## Abstract

The full metric describing a charged, magnetized generalization of the Tomimatsu-Sato (TS) $\delta = 2$ solution is presented in a concise explicit form. We use it to investigate some physical properties of the solution; in particular, we point out the existence of naked ring singularities in the hyperextreme TS metrics, the fact previously overlooked by the researchers, and we also demonstrate that the ring singularities can be eliminated by sufficiently strong magnetic fields in the subextreme case, while in the hyperextreme case the magnetic field can move singularities to the equatorial plane.

KEY WORDS: Ernst potentials, Tomimatsu-Sato metrics, magnetic dipole.

## 1. INTRODUCTION

In [1] we have presented an exact asymptotically flat 4-parameter solution of the Einstein-Maxwell equations constructed with the aid of Sibgatullin's method [11] which generalizes the well-known Tomimatsu-Sato $\delta = 2$ metric [3, 4] and is defined by the Ernst complex potentials $\mathcal{E}$ and $\Phi$ [5] of the form[1]

$$
\begin{aligned}
\mathcal{E} &= \frac{A - 2mB}{A + 2mB}, \quad \Phi = \frac{2C}{A + 2mB}, \\
A &= (k^2 x^2 - \sigma y^2)^2 - (k^2 - \sigma)^2 - 2ik^3 axy(x^2 - 1) \\
&\quad -(1 - y^2)[a(k^2 - 2\sigma) + 2qc][a(y^2 + 1) + 2ikxy], \\
B &= kx[k^2(x^2 - 1) + \sigma(1 - y^2)] - iy(1 - y^2)[a(k^2 - 2\sigma) + 2qc], \\
C &= k^2(x^2 - 1)(kqx + icy) + (1 - y^2)\{k(ac + q\sigma) \\
&\quad -iy[aq(k^2 - 2\sigma) + c(2q^2 - \sigma)]\}, \\
\sigma &\equiv c^2/(m^2 - a^2 - q^2), \quad k \equiv \sqrt{m^2 - a^2 - q^2 + \sigma}, \quad (1)
\end{aligned}
$$

$x$, $y$ being the generalized spheroidal coordinates, and $m$, $a$, $q$, $c$ being four arbitrary real parameters representing the mass, angular momentum, charge and magnetic dipole moment.

The expressions for the corresponding metric functions $f$ and $\gamma$ entering the axisymmetric line element

$$
\begin{aligned}
\mathrm{d}s^2 &= k^2 f^{-1}\Big[e^{2\gamma}(x^2 - y^2)\Big(\frac{\mathrm{d}x^2}{x^2 - 1} + \frac{\mathrm{d}y^2}{1 - y^2}\Big) \\
&\quad + (x^2 - 1)(1 - y^2)\mathrm{d}\varphi\Big] - f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2, \quad (2)
\end{aligned}
$$

have been given in [1] only in terms of the polynomials $A$, $B$ and $C$, while for the remaining function $\omega$ we have been forced to give a rather cumbersome expression because of not having at hand a good strategy for the simplification of complicated coefficients involving four independent parameters.

Recently notwithstanding we have succeeded in getting very concise expressions for $f$ and $\gamma$, and a by far simpler expression for $\omega$ than in Ref. 1. The main objective of this paper, therefore, will be the presentation of the

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[1] In what follows we write $\sigma$ instead of $\sigma^2$ used in Ref. 1 to underline the fact that $\sigma$ may assume both positive and negative values. Throughout the paper units are used in which the gravitational constant and the velocity of light are equal to unity

simplest metric for the exterior field of a charged, magnetized, spinning mass in a concise explicit form most suitable for its analysis and possible applications. Besides, we shall analyse some physical properties of the metric, such as, e.g., the multipole moments, singularities and limits. We shall demonstrate, in particular, that the superextreme TS solutions do have naked ring singularities (contrary to the statement made in [6] on the absence of such singularities) which are located outside the equatorial plane, but can be moved to the equatorial plane by an appropriate choice of the magnetic dipole parameter.

## 2. THE METRIC FUNCTIONS AND LIMITING CASES

The first step in our search for concise expressions of the metric coefficients $f$, $\gamma$ and $\omega$ corresponding to the solution (1) was based on Yamazaki's idea [6, 7] to represent these coefficients in terms of the quantities $x^2 - 1$ and $1 - y^2$. Posterior major simplifications have come from the analysis of the factor structure of the metric coefficients, and in that work the papers by Hoenselaers [8] and Perjés [9] have been our guides. A tedious but straightforward algebra with the use of the Mathematica computer programme [10] has finally led us to the following elegant expressions for $f$, $\gamma$ and $\omega$:

$$
\begin{aligned}
f &= \frac{E}{D}, \quad e^{2\gamma} = \frac{E}{k^8(x^2 - y^2)^4}, \quad \omega = -\frac{2(1 - y^2)F}{E}, \\
E &= \{[k^2(x^2 - 1) + \sigma(1 - y^2)]^2 + a[a(d - \sigma) + 2qc](1 - y^2)^2\}^2 \\
&\quad -4k^2(x^2 - 1)(1 - y^2)[k^2a(x^2 - y^2) + 2(a\sigma - qc)y^2]^2, \\
D &= \{(k^2x^2 - \sigma y^2)^2 + 2kmx[k^2(x^2 - 1) + \sigma(1 - y^2)] \\
&\quad + a[a(d - \sigma) + 2qc](y^4 - 1) - d^2\}^2 \\
&\quad +4y^2\{k^3ax(x^2 - 1) + [a(d - \sigma) + 2qc](kx + m)(1 - y^2)\}^2, \\
F &= 4k^2(x^2 - 1)[k^2a(x^2 - y^2) + 2(a\sigma - qc)y^2] \\
&\quad \times \{kmx[k^2(x^2 + 1) - \sigma(y^2 + 1)] + k^2x^2(2m^2 - q^2) - \sigma dy^2\} \\
&\quad -\{[k^2(x^2 - 1) + \sigma(1 - y^2)]^2 + a[a(d - \sigma) + 2qc](1 - y^2)^2\} \\
&\quad \times \{2k^2qc(x^2 - y^2) + (1 - y^2)[ad(2kmx + 2m^2 - q^2) \\
&\quad -(a\sigma - 2qc)(2kmx + m^2 + a^2)]\}, \\
d &\equiv m^2 - a^2 - q^2.
\end{aligned}
\tag{3}
$$

Note that the above expression for $F$ is five times shorter than the re-

spective expression in Ref. 1! Together with the formulae for $E$ and $D$ it defines explicitly the simplest metric able to describe the exterior field of a charged, magnetized, spinning mass.

Some properties of the solution (1)-(3) can be better seen if its relativistic Simon's multipole moments [11] are given. Below we write out the first four moments obtainable from (1) with the aid of the Hoenselaers-Perjés procedure [12], $M_i$ describing the distribution of the mass ($\text{Re}M_i$) and angular momentum ($\text{Im}M_i$), and $Q_i$ describing the electric ($\text{Re}Q_i$) and magnetic ($\text{Im}Q_i$) fields:

$$
\begin{aligned}
&M_0 = 2m, \quad M_1 = 4ima, \\
&M_2 = -2m(m^2 + 3a^2 - q^2 - \sigma), \\
&M_3 = -8ima(m^2 + a^2 - q^2 - \sigma), \\
&Q_0 = 2q, \quad Q_1 = 2i(c + 2aq), \\
&Q_2 = -2[2ac + q(m^2 + 3a^2 - q^2 - \sigma)], \\
&Q_3 = -2i[(m^2 + a^2 - q^2 - \sigma)(c + aq) + 2a^2c], \quad (4)
\end{aligned}
$$

whence the asymptotic flatness of the solution follows immediately, as well as the physical interpretation of the parameters $m$, $a$, $q$ and $c$ as determining, respectively, the total mass, total angular momentum per unit mass, total charge and magnetic dipole moment of the source. The latter four physical quantities are not restricted anyhow, so that the solution is equally applicable to both the sub- and superextreme cases which correspond to the real and pure imaginary values of $k$, respectively.

Let us consider now the limiting cases of the solution (1)-(3).

a) The limit $q = c = 0$ leads to the Tomimatsu-Sato $\delta = 2$ metric [3, 4].

b) Another well-known limit is Bonnor's two-parameter solution [13] for a static massive magnetic dipole ($q = a = 0$).

c) The case $c = 0$ corresponds to the charged version of the Tomimatsu-Sato $\delta = 2$ solution constructed by Ernst [14].

d) When $q = 0$, one arrives at the Manko-Ruiz solution [15] which is a stationary generalization of the Bonnor metric [13].

e) When the parameters happen to satisfy the relation

$$
m^2 - a^2 - q^2 - \sigma = 0, \quad (5)
$$

the solution can be interpreted as a specific electromagnetic generalization of the Kerr metric [16] different from the Kerr-Newman spacetime [17] if $\sigma \neq 0$.

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f) The limit $a = 0$ is of special interest. It provides an exact three-parameter analog to the approximate solution used by Bonnor [18] for his analysis of the dragging of inertial frames by a charged massive magnetic dipole. In view of the potential physical importance of this particular solution we write it out explicitly:

$$
\begin{aligned}
f &= \frac{E}{D}, \quad e^{2\gamma} = \frac{E}{k^8(x^2 - y^2)^4}, \quad \omega = \frac{4qc(1 - y^2)F}{E}, \\
E &= [k^2(x^2 - 1) + \sigma(1 - y^2)]^4 - 16k^2q^2c^2y^4(x^2 - 1)(1 - y^2), \\
D &= [(k^2(x^2 - 1) + \sigma(1 - y^2)]^2[k^2(x^2 + 1) - \sigma(y^2 + 1) + 2kmx]^2 \\
&\quad + 16q^2c^2y^2(kx + m)^2(1 - y^2)^2, \\
F &= [k^2(x^2 - 1) + \sigma(1 - y^2)]^2[k^2(x^2 - 1) + (2kmx + k^2 + m^2)(1 - y^2)] \\
&\quad + 4k^2y^2(x^2 - 1)(kmx + m^2 - q^2)(k^2x^2 - \sigma y^2 + kmx), \\
k &\equiv \sqrt{m^2 - q^2 + \sigma}, \quad \sigma \equiv \frac{c^2}{m^2 - q^2}.
\end{aligned}
\tag{6}
$$

The total angular momentum of this solution is equal to zero, as well as all its higher rotational multipole moments. At the same time, the metric has a non-vanishing $\mathrm{d}t\mathrm{d}\varphi$ term characterized by the coefficient $\omega$! According to Bonnor [18], this effect of frame-dragging by a charged, massive magnetic dipole is due to the Poynting vector which produces flows of energy in the equatorial plane where the frame-dragging occurs.

Note that the above metric cannot be obtained from the Bonnor magnetic dipole solution [13] by application of the Kramer-Neugebauer charging transformation [19] since the latter transformation generates a specific non-vanishing angular momentum which depends on the parameters of charge and magnetic dipole moment [19, 20].

As a final remark concluding this section let us point out that the potentials (1) corresponding to the pure imaginary values of $k$ belong to the Chen-Guo-Ernst family of electrovacuum hyperextreme solutions [21].

## 3. SINGULARITIES AND STATIONARY LIMIT SURFACE

The structure of singularities of the electrovacuum rational function solutions has some similar as well as distinctive features compared with the pure vacuum case, but in both cases the singularities arise as solutions of the

equation

$$A + 2mB = 0. \tag{7}$$

For the real-valued $k$ (the subextreme case) the solution (1)-(3), similar to the TS $\delta = 2$ solution, has two singular points on the symmetry axis, $x = 1$, $y = \pm 1$, and besides may have a naked ring singularity in the equatorial plane. However, if the latter ring singularity is inevitably present in the subextreme TS $\delta = 2$ solution, it is not necessarily the case for the charged, magnetized TS $\delta = 2$ solution (1)-(3) where the ring singularity can be eliminated by the magnetic field.

In the following three diagrams we have shown the location of singularities for particular values of the parameters of the subextreme TS $\delta = 2$ solution (Figure 1.i) and of the solution (1)-(3) (Figures 1.ii,iii). In addition we have plotted there the shape of the stationary limit surface for each case (on this surface defined by the equation

$$E = 0 \tag{8}$$

the time-like Killing vector becomes a null vector). Note that all figures have been plotted in the Weyl-Papapetrou cylindrical coordinates $(\rho, z)$ introduced via the formulae

$$x = \frac{1}{2k}(r_+ + r_-), \quad y = \frac{1}{2k}(r_+ - r_-), \quad r_\pm \equiv \sqrt{\rho^2 + (z \pm k)^2}. \tag{9}$$

On can see (Figure 1.i) that all the three singularities of the TS $\delta = 2$ solution lie on the stationary limit surface (this is of course the general property of the stationary vacuum solutions). In the presence of the electromagnetic field with the value of the magnetic dipole parameter less than the critical one, the naked ring singularity is still present but it is not already located on the stationary limit surface (Figure 1.ii). When the parameter $c$ exceeds the critical value (which is approximately 0.8254 for the chosen values of the parameters $m$, $a$ and $q$), the ring singularity disappears (Figure 1.iii).

Turning now to the hyperextreme case characterized by the pure imaginary values of $k$, it should be first of all remarked that Yamazaki's statement ([6], p. 2505) about the absence of naked ring singularities in the hyperextreme TS metrics is erroneous. It is true that the hyperextreme TS solutions have no singular points in the equatorial plane; however, the singularities arise outside the equatorial plane. Figure 2.i shows that the stationary limit
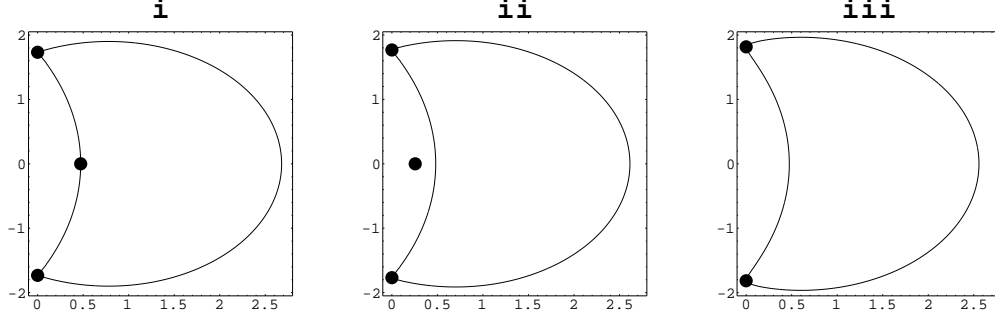
Figure 1: Ergosphere and singularities in the subextreme case. The particular choice of the parameters is: i) m = 2, a = 1, q = c = 0; ii) m = 2, a = 1, q = 0.2, c = 0.7; iii) m = 2, a = 1, q = 0.2, c = 1.

surface of the hyperextreme TS $\delta = 2$ solution is a torus with two ring singularities on it. By adding the electromagnetic field, the singularities can be brought closer to each other (Figure 2.ii). Lastly, one can see that a sufficiently strong magnetic field moves the singularities to the equatorial plane (Figure 2.iii), eliminating one of them.

## 4. THE MAGNETIC POTENTIAL

The electric and magnetic fields in the solution (1)-(3) are described, respectively, by the $A_4$ and $A_3$ components of the electromagnetic four-potential. The component $A_4$ is simply the real part of the Ernst potential $\Phi$ defined by (1). The determination of $A_3$ is most simple via the construction of Kinnersley's complex scalar potential $\Phi_2$ [22]. The details of the derivation of the latter potential in Sibgatullin's method can be found, e.g., in Ref. 23; hence, in what follows we shall restrict ourselves to only writing out the resulting expression for $\Phi_2$:

$$
\begin{aligned}
\Phi_2 &= \frac{2G}{A + 2mB} - 2iq, \\
G &= k^2(x^2 - 1)\{(1 - y^2)[c(kx + 3m) + iy(ac + q\sigma)] + 2kaqx \\
&\quad + iqy[k^2(x^2 + 1) + 2kmx - 2\sigma]\} + (1 - y^2)\{2(kx + m) \\
&\quad \times [aqd + c(2m^2 + q^2)] - [aq(d - \sigma) + c(2q^2 - \sigma)] \\
&\quad \times [(kx + m)(1 - y^2) + 2iay] + 2m[c(\sigma - 2a^2 - q^2) - aq\sigma]\}. \quad (10)
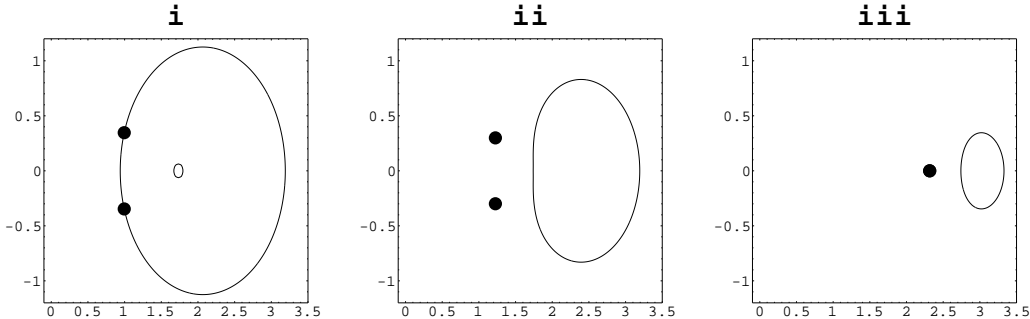\end{aligned}
$$

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Figure 2: Ergosphere and singularities in the hyperextreme case. The particular choice of the parameters is: i) m = 1, a = 2, q = c = 0; ii) m = 1, a = 2, q = 0.2, c = 1.1; iii) m = 1, a = 2, q = 0.2, c = 2.3.

The magnetic potential $A_3$ is determined consequently as the real part of $\Phi_2$. On the four diagrams (Figures 3.i-iv) we have plotted the magnetic lines of force for different particular parameter sets which cover both the sub- and superextreme cases.

## 5. CONCLUSIONS

Therefore, we have succeeded in giving concise explicit expressions for all the metric coefficients defining the charged, magnetized generalization of the Tomimatsu-Sato $\delta = 2$ solution, and we have analysed some physical properties of the new electrovac solution. The latter has been shown to have several well-known limits, as well as some new limits among which the three-parameter solution for a charged, massive magnetic dipole is probably of special interest.

The study of singularities of the solution (1)-(3) has enabled us to correct an old erroneous belief that the hyperextreme TS solutions have no ring singularities. At the same time we have shown that a magnetic field can eliminate these naked ring singularities in the subextreme case, and move them to the equatorial plane from outer regions in the superextreme case.

The existence of naked ring singularities located outside the equatorial plane in the hyperextreme TS solutions probably makes the latters not quite appropriate for modelling the exterior fields of single infinitesimally thin rela-
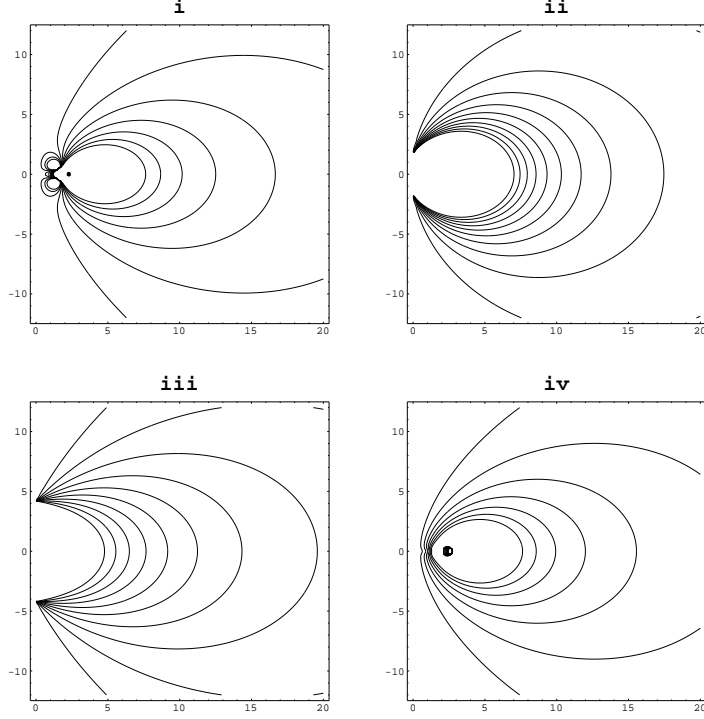
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Figure 3: Magnetic lines of force. The particular choice of the parameters is: i) m = 1, a = 2.5, q = 0, c = 1; ii) m = 2, a = 1, q = -0.5, c = 1; iii) m = 1, a = 0, q = -0.2, c = 4; iv) m = 1, a = 1.5, q = -0.2, c = 2.5.

tivistic disks where the Neugebauer-Meinel (global) solution [24, 25] and the hyperextreme Kerr solution [16] seem to be the only possibilities to describe such objects (however, the hyperextreme TS metrics most likely could describe the fields of superposed disks). At the same time, there is of course no any problem in interpreting the TS metrics as describing the exterior fields of deformed masses, and besides we hope that the subclasses of the magnetized hyperextreme TS solutions whose singularities are located exclusively in the equatorial plane still could be considered as good candidates even for the description of the exterior fields of single magnetized thin disks.

## ACKNOWLEDGEMENTS

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